

Chif and G.A. seminar / 2/25/19.
O. Dess.

Twisted harmonic maps and self-duality.

Goal: use harmonic maps to solve an equation in gauge theory.

X - compact Riemann surface

N - target mfd (possibly noncompact).

Let $\rho: \pi_1(X) \rightarrow \text{Isom}(N)$.

Defn: A twisted harmonic map is a ρ -equivariant harmonic map

$$s: \hat{X} \rightarrow N.$$

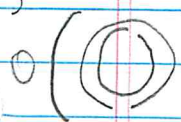
$$\begin{array}{ccc} \underline{E}: & \hat{X} & \xrightarrow{\text{Id.}} \tilde{X} = N. \\ & \downarrow & \downarrow \\ & X & \xrightarrow{\quad} X \end{array}$$

Prop: If ρ acts freely, then $\tilde{X} \cong N/\rho$.
 $\cong \text{hom } s: X \rightarrow N/\rho.$

$\mathbb{R}$

$\downarrow$

s'



Equivalently: Define the fiber bundle

$$O((\odot)_{+1}) N \rightarrow \tilde{X} \times_{\rho} N = N \times$$

$\downarrow$
 X

Flat bundle w/ fibers isom. to N .

Vertical tan. bundle $(T_v N)_{x_0} \simeq T_{x_0} N$.

Given $s: X \rightarrow N$ section, write

$$D_s: (TX)_x \rightarrow (T_v N)_{s(x_0)} ??$$

$$E[s] = \frac{1}{2} \int |D_s|^2 dV dx.$$

Twisted Huis = harmonic sections of N

Henceforth, let $N = \mathbb{H}^3$.

1. Hyperboloid model:

$$\mathbb{H}^3 = \{x \in \mathbb{R}^{3,1} \mid |x|_m = 1\}$$

$$|x|_m^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

with induced (diem.) metric.

$$\text{Isom}(\mathbb{H}^3) = SO(3,1) \cdot (\text{Lorentz group})$$

dim $\mathbb{H}^3 = 6$.

2. Ball model:

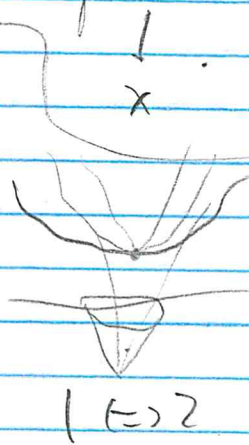
$$\mathbb{H}^3 = \mathbb{D}^3 \subset \mathbb{R}^3$$

$$g = \frac{4}{(1-r^2)^2} g_{\text{Eucld}}$$

$$\bar{\mathbb{H}}^3 = \mathbb{H}^3 \cup \mathbb{CP}^1$$

conformal compactif.

(upper-half-space model)



3. Matrix model:

$$\text{Herm}^{2 \times 2} = \{ H \in \mathbb{C}^{2 \times 2} \mid H^\dagger = H \}$$

$$\mathbb{H}^3 = \{ H \in \text{Herm}^{2 \times 2} \mid \det H = 1 \}$$

$$\text{Nat. action by } \text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C}) / \pm 1$$

$$H \mapsto g H g^\dagger$$

(left-inv.) Metric: $|dH|_H^2 = \frac{1}{2} \text{Tr}(H^{-1} dH)^2$ on $\mathbb{H}^3$.

$$\Rightarrow \boxed{\mathbb{H}^3 = \text{SL}(2, \mathbb{C}) / \text{SU}(2)}$$

$$\mathfrak{sl}_2 \mathbb{C} = \mathfrak{su}_2 \oplus i \mathfrak{su}_2$$

($\Rightarrow$ neg. curved
using Killing, actually!)

To see equivalence 1. $\Leftrightarrow$ 3.

Before the above metric on $\text{Herm}^{2 \times 2}$

$$|H|_m^2 = \det H$$

$$H = \begin{pmatrix} x+y & z \\ \bar{z} & x-y \end{pmatrix}, \quad z \in \mathbb{C}$$

$$\det H = x^2 - y^2 - |z|^2$$

Agrees with above metric since both are left-inv.

Explicitly: if $H \in (\mathbb{H}^3)_H$ then $\text{Tr}(H^{-1} dH) = \log \det H$

$\Rightarrow$ eigs λ_1, λ_2 of $H^{-1} dH$ sat. $\lambda_1 + \lambda_2 = 0$.

$$\Rightarrow \frac{1}{2} \text{Tr}(H^{-1} dH)^2 = \frac{1}{2} (\lambda_1^2 + \lambda_2^2) = \lambda_1^2 \Rightarrow \det(H^{-1} dH) = \det(dH)$$

✓

Reason for using matrix model:

$TH^3 \otimes \mathbb{C}$ has a natural flat structure
(right) - (global invariant frame).
 $H^3 = \mathrm{PSL}(2, \mathbb{C}) / \mathrm{SO}(3)$

$TH^3 = \mathrm{SO}(3)$ - right - invt sections
of $T\mathrm{PSL}(2, \mathbb{C}) \perp T\mathrm{SO}(3)$.

$\Rightarrow TH^3 \otimes \mathbb{C} = \mathrm{SO}(3)$ - right - invt sections
of $T\mathrm{PSL}(2, \mathbb{C})$
 $\cup$

{right - invt global frame}
should write down explicitly.

Given $p: \pi_1(X) \rightarrow \mathrm{PSL}(2, \mathbb{C}) = \mathrm{Isom}(H^3)$,

form $\mathcal{H} = \tilde{X} \times_p H \rightarrow X$.

Then $T\mathcal{H} \otimes \mathbb{C}$ has a canonical flat
 $\mathrm{PSL}(2, \mathbb{C})$ - connection.

(projection to real part is Levi-Civita).

any $m^{2 \times 2} \in H^{2 \times 2}$ is NOT equiv to TH^3 . but
 $m^{2 \times 2} \cdot H^{-1}$ is.

doesn't / doesn't the frame just have to be

$(F_i)_{ij} = g^{-1} \cdot g_i \cdot g_j^{-1}$? How the fuck is it well-defined?

but small
problem of
being perpendicular
on working
frames.

Recall self-duality eqns in dim 4:

$$F = F_A^{\text{be}} = 2A + A \wedge A = \frac{1}{2} F_{ij} dx^i \wedge dx^j$$

$$F_{ij} = 2i d_j - 2j d_i + [A_i, A_j]$$

S.D.: $F = *F \Leftrightarrow$

$$F_{12} = F_{34}$$

$$F_{13} = F_{42}$$

$$F_{14} = F_{23}$$

~~normal~~ normal cross.

$M = \mathbb{R}^4$, consider forms

$$A = \underbrace{A_0 dx_1 + A_1 dx_2}_{A_0} + \underbrace{\phi_1 dx_3 + \phi_2 dx_4}_{A_1}$$

with A_0, ϕ_i indep. of x_3, x_4 .

$$F_A = F_{A_0} + d_{A_0} A_1 + A_0 \wedge A_1$$

$$= F_{A_0} + d_{A_0} d_1 dx_3 + d_{A_0} d_2 dx_4 + [\phi_1, \phi_2] dx_3 \wedge dx_4$$

S.D.: $(F_{A_0})_{12} = [\phi_1, \phi_2]$

$$\nabla_1 \phi_1 = -\nabla_2 \phi_2$$

$$\nabla_2 \phi_1 = \nabla_1 \phi_2$$

let $\phi = \phi_1 - i\phi_2$, $A_0 \rightarrow A$.

$\Leftrightarrow F_{12} = \frac{1}{2} i [\phi, \phi^*]$

$$(\nabla_1 + i\nabla_2)\phi = 0$$

$P \rightarrow \text{SO}(3) \text{ - bundle.}$

$M = X \times \mathbb{R}^2$. let $\bar{\Phi} = \frac{1}{2} \phi dz \in \Omega^{1,0}(\text{ad } P \otimes \mathbb{C})$.

Get:

$$\boxed{\begin{aligned} (*) \quad & F + [\bar{\Phi}, \bar{\Phi}^*] = 0. \\ & \bar{\partial}_A \bar{\Phi} = 0. \end{aligned}}$$

Depends only on conf'l str. z on X .

Nitchin shows: for any stable hol. v.b. $E \rightarrow X$,
w/ hol. Higgs field $\bar{\Phi}$, has a ^{compatible} connection A
solving Nitchin eqs.

alternatively: let $\Psi = \bar{\Phi} + \bar{\Phi}^* \in \Omega^1(\text{ad } P)$

(*) $\Leftrightarrow$ (i) $F_A + \frac{1}{2} [\Psi, \Psi] = 0$

(ii) $d_A \Psi = 0$

(iii) $d_A^* \Psi = 0.$

$\downarrow$
actually real
stuff.

Notice: if we define the $\text{PSU}(2,1)$ - connection

$d_A + \gamma \in \Omega^1(\text{ad } P)$ then

$$\begin{aligned} (d_A + \gamma)^2 &= d_A^2 + d_A \gamma - \gamma \wedge d_A \\ &\quad + \gamma \wedge d_A + \gamma \wedge \gamma. \\ &= (F_A + d_A \gamma + \gamma \wedge \gamma) \in \Omega^2(\text{ad } P). \end{aligned}$$

$\cap \mathcal{M}(\text{ad } P)$

$\Rightarrow$ a flat $PSL(2, \mathbb{C})$ connection on P^c
 $dA + \dots$
 solves (i) and (ii).

Given a flat $PSL(2, \mathbb{C})$ -bundle $(p: \pi_1(X) \rightarrow PSL(2, \mathbb{C}))$
 we can form the H^3 bundle $\mathcal{H}$ as above.

if we any flat $PSL(2, \mathbb{C})$ on P^c has one

Given a section s , we obtain a flat
 (complex) connection on $s^*(T\mathcal{H} \otimes \mathbb{C}) := P^c$ by pullback:

$$d + A_s + \psi_s, \text{ sat. (i) \& (ii).}$$

Prop. (A_s, ψ_s) sat. (iii) $\Leftrightarrow s$ is a
 twisted harmonic map.

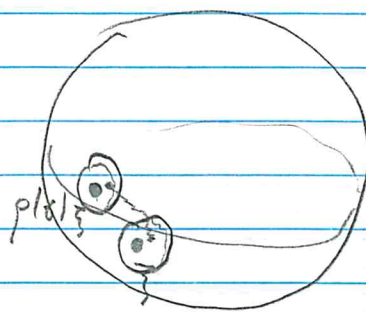
$\Rightarrow$ Theorem: any ired. flat $PSL(2, \mathbb{C})$ -connection
 is equiv. to a solution of Hitchin's eqs.

(NAUT.)

Def: Ellis-Samson: $|D_s| < C$.

ired $\Leftrightarrow \forall \xi \in \mathbb{CP}^1, \exists$

$[\delta] \in \pi_1(X)$ s.t. $p([\delta]\xi) \neq \xi$.



But $d(x_0, s(x_0)) < C \Rightarrow d(s(x_0), s([\delta]x_0)) < C \cdot C$.

$\nearrow$
 X